$$a^* (h) \to a / 5 \quad \text{for} \quad h \to \infty, \qquad a^* (h) \to a/6 \quad \text{for} \quad h \to 0$$

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BOUNDARY LAYER IN THE PROBLEM OF LONGITUDINAL MOTION OF A CYLINDER IN A VISCOPLASTIC MEDIUM

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Construction of the boundary layer by the method of variation is used for investigating the principle of selecting the unique solution for a perfectly plastic medium by transition to it from a viscoplastic medium with the viscosity coefficient tending to zero.

Let an infinitely long cylinder move along its axis in a viscoplastic medium at constant velocity. The velocity field of particles of a viscoplastic medium induced by the cylinder motion in a system of coordinates x, y, z (with the cylinder axis along the z-coordinate and its cross section ω lying in xy-plane) is of the form $u = (0, 0, u (x, y))$. It was shown in [1] that $u (x, y)$ minimizes functional

$$I_1 (w) = \int\limits_{R^2 \setminus \omega} \left[\frac{\mu}{2} | \nabla w |^2 + \tau_0 | \nabla w | \right] dw - Fw |_{d\omega}, \quad w |_{d\omega} = \text{const}$$

where μ and τ_0 are, respectively, the viscosity coefficient and the yield point of the medium and F is the longitudinal force moving the cylinder. The velocity of the cylinder can be determined when force F is specified. It is $u (x, y)$ over $\partial\omega$. If the cylinder velocity is u_0, then $u (x, y)$ minimizes functional

$$I_2 (w) = \int\limits_{R^2 \setminus \omega} \left[\frac{\mu}{2} | \nabla w |^2 + \tau_0 | \nabla w | \right] dw, \qquad w |_{\partial\omega} = u_0 \tag{1}$$

and the force necessary for producing such motion is determined by formula

$$u_0 F = I_2 (u) + \int\limits_{R^2 / \omega} \frac{\mu}{2} | \nabla u |^2 d\omega$$

Below we consider the particular case in which the cylinder velocity is specified.

It was shown in [1] that in the case of convex region ω the motion of the cylinder is possible if $F > \tau_0 \operatorname{mes} \partial\omega$.

The limit case of viscoplastic medium is a perfectly plastic medium for which $\mu = 0$. In this case stationary motion is only possible for $F = \tau_0 \operatorname{mes} \partial\omega$. The velocity distribution of stationary motion is then

$$u(x, y) = \begin{cases} u_0, & (x, y) \in \partial\omega \\ 0, & (x, y) \in R^2 \setminus \omega \end{cases}$$

The motion of a system of cylinders with parallel axes in a perfectly plastic medium was considered in [2]. We shall illustrate the results obtained there on the example of two cylinders of cross sections ω_1 and ω_2 (see Fig. 1). The pattern of velocity field of such system is determined by the relative position of ω_1 and ω_2. If $\operatorname{mes} \partial (\omega_1 \cup \omega_2) < \operatorname{mes} \partial (\omega_1 \cup \Omega \cup \omega_2)$, the cylinders move independently from each other at velocities u_1 and u_2, while the medium is stationary. Conversely, if $\operatorname{mes} \partial (\omega_1 \cup \omega_2) > \operatorname{mes} \partial (\omega_1 \cup \Omega \cup \omega_2)$, region $\omega_1 \cup \Omega \cup \omega_2$ moves at constant velocitiy while the remaining part of the medium is stationary. In the limit case, when $\operatorname{mes} \partial (\omega_1 \cup \omega_2) = \operatorname{mes} \partial (\omega_1 \cup \Omega \cup \omega_2)$, function $u(x, y)$ is of the form

$$u(x, y) = \begin{cases} u_1, & (x, y) \in \partial\omega_1 \\ u_2, & (x, y) \in \partial\omega_2 \\ u_0, & (x, y) \in \Omega \\ 0, & (x, y) \in R^2 \setminus (\omega_1 \cup \Omega \cup \omega_2) \end{cases} \tag{2}$$

$$u_1 \geqslant u_0, \qquad u_2 \geqslant u_0, \qquad u_0 \geqslant 0$$

Thus various configurations of the velocity field are possible here. Note that in the case of motion of the same system of cylinders in a viscoplastic medium the velocity configuration is uniquely determined. Hence the natural question: which of the configurations of the velocity field is the limit one for $\mu \to 0$. To answer that question it is necessary to analyze the properties of velocity fields in the above problem for a viscoplastic medium whose viscosity is low. It is shown below that the choice of the limit configuration of the velocity field is determined by energy dissipation in the boundary layer which develops in the neighborhood of the discontinuity line of function $u(x, y)$ in formula (2). The scheme for the construction of the velocity field in the boundary layer proposed below is based on the asymptotic method of variation. Various examples of the use of that method appear in [3 – 5].

Let us now consider functional (1) and pass in it to dimensionless variables. We normalize x and y with respect to length L of the contour $\partial\omega$ and the velocity field with respect to u_0. I_2 then becomes

$$I_2(w) = \tau_0 u_0 L \int_{R^2/\omega} \left[\frac{\varepsilon}{2} |\nabla w|^2 + |\nabla w|\right] d\omega, \qquad w|_{\partial\omega} = 1 \tag{3}$$

$$\varepsilon = \mu u_0 / \tau_0 L \ll 1 \tag{4}$$

The main object of this work is to determine with specific accuracy the approximate function which would minimize (3). Thus, if u_ε is a function which minimizes (3), then

$$I_2(u_\varepsilon) = \tau_0 u_0 L\,[1 + c_1 f_1(\varepsilon) + c_2 f_2(\varepsilon) + \ldots] \tag{5}$$

$$\lim_{\varepsilon\to 0} \frac{f_n(\varepsilon)}{f_{n-1}(\varepsilon)} = 0 \qquad (n = 2, 3, \ldots)$$

where the expansion is asymptotic and function $f_i(\tau)$ is independent of the contour geometry. We call function u_ε^k an approximation of order k, if

$$I_2(u_\varepsilon^k) - I_2(u_\varepsilon) = \tau_0 L u_0\, g_{k+1}(\varepsilon)$$

$$\overline{\lim_{\varepsilon\to 0}} \left| \frac{g_{k+1}(\varepsilon)}{f_{k+1}(\varepsilon)} \right| < \infty$$

It can be shown [3] that

$$\int\limits_{R^2 \setminus \omega} |\nabla u_\varepsilon^k - \nabla u_\varepsilon|^2 d\omega \leqslant \frac{4}{\varepsilon}\, g_{k+1}(\varepsilon) \qquad (k = 1, 2, \ldots) \tag{6}$$

The inequality (6) represents an estimate of the difference between the true velocity field and its approximation.

We shall call the first order approximation of u_ε the boundary layer. Note that, when $\lim_{\varepsilon\to 0}(g_2(\varepsilon)/\varepsilon) \neq 0$, the boundary layer does not, generally speaking, give any qualitative idea of the behavior of u_ε. Hence the investigation of u_ε requires the consideration of higher order approximations.

Let us describe the procedure of approximation derivation. We point out that the most important step is the estimation of the lower limit of functional (3). We pass in functional (3) to orthogonal curvilinear coordinates (s, n) such that the form of ω is mapped into a half-band $0 \leqslant s \leqslant a$ and $n \geqslant 0$; $n = 0$ defines the boundary of ω and

$$I_2 = \tau_0 L u_0 \int_0^a\!\!\int_0^\infty \left\{ \frac{\varepsilon}{2}\left[A^2(s,n)\left(\frac{\partial u}{\partial n}\right)^2 + C^2(s,n)\left(\frac{\partial u}{\partial s}\right)^2 \right] + \left[A^2(s,n)\left(\frac{\partial u}{\partial n}\right)^2 + C^2(s,n)\left(\frac{\partial u}{\partial s}\right)^2 \right]^{1/2} \right\} \Delta\, ds\, dn$$

$$\Delta = \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial n} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial n} \right|$$

We introduce functional

$$I_2^*(u) = \tau_0 u_0 L \int_0^a\!\!\int_0^\infty \left[\frac{\varepsilon}{2} A^2 \left(\frac{\partial u}{\partial n}\right)^2 + \left| A \frac{\partial u}{\partial n} \right| \right] \Delta\, ds\, dn$$

Obviously $I_2 \geqslant I_2^*$ and, consequently, $\inf I_2 \geqslant \inf I_2^*$. Let u_ε^* minimize functional I_2^*. Then for $\inf I_2$ we have the upper and lower bound estimate

$$I_2(u_\varepsilon^*) \geqslant \inf I_2(u) \geqslant I_2^*(u_\varepsilon^*) \tag{7}$$

Note that, if lines of level of function u_ε and lines orthogonal to these are taken as the coordinates, the inequalities (7) become equalities. Hence it is possible to propose the following method of approximation derivation. We determine functional I_2^* in any arbitrary system of coordinates and find for it function u_ε^*. We then pass to the new system of coordinates defined by level lines of u_ε^* and lines orthogonal to it. We again determine functional I_2^* in the new system, find for it function u_ε^*, and then repeat preceding constructions. It is obvious that the accuracy of (7), which determines the order of approximation, increases with each successive step. It is important to note that the determination of u_ε necessitates the solution of a differential equation in partial derivatives, while the determination of u_ε^* requires the solution of an ordinary differential equation. The construction described above can be obtained also in the case of nonorthogonal coordinates. However, it should be borne in mind that the accuracy of approximation depends on the extent of nonorthogonality of coordinates.

One of the possible system of coordinates for the considered here problem is the following:

$$x = x(s) + y'(s)\,n, \qquad y = y(s) - x'(s)\,n \qquad (0 \leqslant s \leqslant 1, \quad n \geqslant 0) \tag{8}$$

where $x = x(s)$ and $y = y(s)$ constitute the natural equation of contour $\partial\omega$. We assume that functions $x(s)$ and $y(s)$ are fairly smooth and that the contour curvature $k(s)$ is strictly positive. The equation for I_2 in variables (8) is of the form

$$I_2(u) = \tau_0 L u_0 \int_0^1 \int_0^\infty \left\{ \frac{\varepsilon}{2} \frac{1}{(1+kn)^2} \left[\left(\frac{\partial u}{\partial s}\right)^2 + (1+kn)^2 \left(\frac{\partial u}{\partial n}\right)^2 \right] + \frac{1}{1+kn} \left[\left(\frac{\partial u}{\partial s}\right)^2 + (1+kn)^2 \left(\frac{\partial u}{\partial n}\right)^2 \right]^{1/2} \right\} (1+kn)\, ds\, dn$$

$$I_2^* = \tau_0 L u_0 \int_0^1 \int_0^\infty \left\{ \frac{\varepsilon}{2} \left| \frac{\partial u}{\partial n} \right|^2 + \left| \frac{\partial u}{\partial n} \right| \right\} (1+kn)\, ds\, dn$$

Function u_ε^* is determined by formula

$$u_\varepsilon^* = 1 - \int_0^n \frac{k(s)(\gamma(s) - t)}{\varepsilon(1 + k(s)\,t)}\, dt \qquad \left(\varepsilon = k(s) \int_0^{\gamma(s)} \frac{\gamma(s) - t}{1 + k(s)\,t}\, dt \right) \tag{9}$$

where $\gamma(s)$ is determined by the equation shown in parentheses. Formula (9) makes it possible to derive function

$$v_\varepsilon^* = \begin{cases} 1 - \frac{k}{\varepsilon} n \left(\sqrt{\frac{2\varepsilon}{k}} - \frac{n}{2} \right), & 0 \leqslant n \leqslant \sqrt{\frac{2\varepsilon}{k}} \\ 0, \ n \geqslant \sqrt{\frac{2\varepsilon}{k}} \end{cases} \tag{10}$$

Function v_ε^* is derived from u_ε^* by expanding it into a Taylor series. A direct check shows that

$$I_2^*(u_\varepsilon^*) - I_2^*(v_\varepsilon^*) = O(\varepsilon) \tag{11}$$

A further direct check shows that

$$I_2|(v_\varepsilon^*) - I_2^*(v_\varepsilon^*) = O(\varepsilon) \tag{12}$$

Note that the right-hand parts of equalities (11) and (12) can be readily estimated with the use of input data of the problem. From (11), (12) and (7) we obtain

$$\inf_w I_2(w) = I_2(v_\varepsilon^*) + O(\varepsilon) = \tau_0 L u_0 \left(1 + \frac{2\sqrt{2}}{3} \sqrt{\varepsilon} \int_0^1 \sqrt{k} ds + O(\varepsilon) \right) \tag{13}$$

Hence

$$f_1(\varepsilon) = \sqrt{\varepsilon}, \qquad c_1 = \frac{2\sqrt{2}}{3}\int_0^1 \sqrt{k}ds$$

In above formulas v_ε^* is the boundary layer for u_ε and $g_2(\varepsilon) = \varepsilon$. The above analysis applies locally, i.e. if part Γ of the contour and region ω_0 bounded by Γ and normals to it at its extreme points (see Fig. 2) is considered, then

$$\inf_w I_2(w) = \tau_0 \operatorname{mes}\Gamma u_0 \left(1 + \frac{2\sqrt{2}}{2}\sqrt{\varepsilon}\int_0^1 \sqrt{k}ds + O(\varepsilon)\right) \tag{14}$$
$$\varepsilon = \frac{\mu u_0}{\tau_0 \operatorname{mes}\Gamma}$$

where $w|_\Gamma = 1$ and w tends to vanish for a point in ω tending to infinity along the normal to Γ. Here and in formula (13) we have introduced the dimensionless variable $0 \leqslant s \leqslant 1$. Let us rewrite formula (14) in the original variables. We have

$$\inf_w I_2(w) = \tau_0 \operatorname{mes} u_0\Gamma + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0 u_0^3}\, M(\operatorname{mes}\Gamma) + O(\mu) \tag{15}$$
$$M(\operatorname{mes}\Gamma) = \int_0^{\operatorname{mes}\Gamma} \sqrt{k(s)}\, ds$$

Estimate (6) implies that
$$\int_{R^2\setminus\omega} |\nabla u_\varepsilon - \nabla v_\varepsilon^*|^2 d\omega < \infty \tag{16}$$

Generally speaking, the integral (16) may not tend to zero when $\varepsilon \to 0$. Consequently, for the determination of function u_ε it is necessary to compute approximations of higher orders.

Let us use function v_ε^* for constructing the coordinate system. We take function $w_\varepsilon^* = 1 - n\sqrt{(2k/\varepsilon)}$ and investigate the lines of its level. The set of lines orthogonal to these lines of level cannot be defined by simple formulas. Because of this we restrict our requirements to the stipulation that the coordinate lines must be orthogonal to within n^2. The above reasoning leads to the following system of coordinates:

$$\begin{aligned} x &= x(s) + y'(s)\frac{n}{\sqrt{k(s)}} + x'(s)\frac{k'(s)}{4k^2(s)}n^2 \\ y &= y(s) - x'(s)\frac{n}{\sqrt{k(s)}} + y'(s)\frac{k'(s)}{4k^2(s)}n^2 \end{aligned} \tag{17}$$

Expressing I_2 in the curvilinear coordinates (17), we obtain

$$\begin{aligned} I_2 = \tau_0 L u_0 \int_0^1\int_0^\infty \Big\{\frac{\varepsilon}{2}\Big[A^2\Big(\frac{\partial u}{\partial n}\Big)^2 + 2C\frac{\partial u}{\partial n}\,\frac{\partial u}{\partial s} + B^2\Big(\frac{\partial u}{\partial s}\Big)^2\Big] + \\ \Big[A^2\Big(\frac{\partial u}{\partial n}\Big)^2 + 2C\frac{\partial u}{\partial n}\,\frac{\partial u}{\partial s} + B^2\Big(\frac{\partial u}{\partial s}\Big)^2\Big]^{1/2}\Big\}\Delta\, ds\, dn \end{aligned} \tag{18}$$

For the functional (18) I_2^* is of the form

$$I_2^* = \tau_0 L u_0 \int_0^1\int_0^\infty \Big[\frac{\varepsilon}{2}\Big(A^2 - \frac{C^2}{B^2}\Big)\Big(\frac{\partial u}{\partial n}\Big)^2 + \sqrt{A^2 - \frac{C^2}{B^2}}\,\Big|\frac{\partial u}{\partial n}\Big|\Big]\Delta\, ds\, dn \tag{19}$$

It is seen that $I_2 \geqslant I_2^*$, hence $\inf I_2 \geqslant \inf I_2^*$. Function u_ε^* which minimizes functional (19) is determined by formula

$$u_\varepsilon^* = \begin{cases} 1 - \frac{1}{\varepsilon} H(n, s), & 0 \leqslant n \leqslant \gamma(s) \\ 0, \quad n \geqslant \gamma(s) \end{cases} \tag{20}$$

$$H(n, s) = \int_0^n \Delta(n', s) \frac{\Phi(\gamma(s), s) - \Phi(n', s)}{\Phi^2(n', s)} dn'$$

Function $\gamma(s)$ is determined by equation

$$\varepsilon = H(\gamma(s), s) \tag{21}$$

where

$$\Phi^2 = \left[A^2(s, n) - \frac{C^2(s, n)}{B^2(s, n)}\right] \Delta^2(s, n)$$

$$\Delta(s, n) = \frac{1}{\sqrt{k(s)}} + n + \frac{k''k - k'^2}{4k^3\sqrt{k}} n^2 + \frac{k'^2}{8k^3} n^3$$

$$A^2 = \frac{1}{\Delta^2}\left[1 + 2n\sqrt{k} + \left(k - \frac{5k'^2}{4k^3} + \frac{k''}{2k^2}\right) n^2 + \frac{2k''k - 3k'^2}{4k^2\sqrt{k}} n^3 + \left\{\left[\left(\frac{k'}{4k^2}\right)'\right]^2 + \frac{k'^2}{16k^2}\right\} n^4\right]$$

$$B^2 = \frac{1}{\Delta^2}\left(\frac{1}{k} + \frac{k'^2}{4k^4} n^2\right)$$

$$C = \frac{1}{\Delta^2}\left[\frac{k'}{4k\sqrt{k}} n^2 + \frac{k'}{8k^2}\left(\frac{k'}{k^2}\right)' n^3\right]$$

To determine $\gamma(s)$ the integrand in (21) must be expanded into Taylor series in powers of n'. The accuracy of $\gamma(s)$ which is required for determining expansion (5) is

$$\gamma(s) = \sqrt{2\varepsilon} + \varepsilon R + \varepsilon\sqrt{\varepsilon} M + \varepsilon^2 N + O(\varepsilon^2\sqrt{\varepsilon})$$

The coefficients R, M and N are expressed in terms of $k(s)$ and its derivatives by using Eq. (21). For instance,

$$R = \frac{1}{3}\left[\sqrt{k} + \frac{5}{2} \frac{k'^2}{k^3\sqrt{k}} - \frac{k''}{k^2\sqrt{k}}\right]$$

Formulas for M and N are not presented owing to their awkwardness. The obtained function u_ε^* (20) is substituted into functionals (18) and (19). Using the expansion of u_ε^* in powers of n and carrying out direct calculations, we find that functionals (18) and (19) differ by the order of ε^2. Thus we obtain the following expansion:

$$\inf I_2 = \tau_0 L u_0 \left\{1 + \frac{2\sqrt{2}}{3}\sqrt{\varepsilon}\int_0^1 \sqrt{kcs}\, ds + \frac{1}{2}\varepsilon\int_0^1 \left[k - \frac{k'^2}{12k^3}\right] ds - \right. \tag{22}$$

$$\left. \frac{53\sqrt{2}}{90}\varepsilon\sqrt{\varepsilon}\int_0^1 \left[k\sqrt{k} + \frac{169}{53}\frac{k'^2}{k^2\sqrt{k}} - \frac{13}{212}\frac{k'^4}{k^6\sqrt{k}} + \frac{1}{53}\frac{k''}{k^4\sqrt{k}}\right] ds + O(\varepsilon^2)\right.$$

Expansion (22) shows that the obtained function u_ε^* is a third order approximation of function u_ε, and estimate (6) is of the form

$$\int_{R^2\setminus\omega} |\nabla u_\varepsilon - \nabla u_\varepsilon^*|^2 \, d\omega \leqslant C\varepsilon$$

Let us revert to the problem of separating out the unique solution of the problem of motion of cylinders in a perfectly plastic medium, using the device of vanishing viscosity. Let us assume that ω_1 and ω_2 (Fig. 1) are convex regions whose boundaries have a positive curvature at all of their points, and that

$$\text{mes}\ \partial\omega_1 + \text{mes}\ \partial\omega_2 = \text{mes}\ \partial(\omega_1 \cup \omega_2 \cup \Omega)$$

Let us further assume that the forces acting on the cylinders depend on μ in such a way that the velocity of the cylinder of cross section ω_i is equal u_i $(i = 1,2)$, where u_i is the same as that appearing in formula (2). Let $u_1 \geqslant u_2$. Obviously $F_i \to \tau_0$ mes ω_i, when $\mu \to 0$ $(i = 1,2)$. If regions ω_1 and ω_2 are identical and $u_1 = u_2$, then $F_1 = F_2$. The solution of the related problem for a perfectly plastic medium is provided by formula (2) for any value of u_0 within the limits $0 \leqslant u_0 \leqslant u_2$. It will be shown that for $\mu \to 0$ solutions of viscoplastic problems converge to the solution of the perfectly plastic problem for $u_0 = u_2$.

Let us assume that the contrary is true. Let there exist a sequence $\mu_i \to 0$ such that solutions of viscoplastic problems converge to the solution (2) for $u_0 < u_2$. Let us consider functional

$$I_3(w) = \int_{R^2\setminus(\omega_1\cup\omega_2)} \left[\frac{\mu}{2}|\nabla w|^2 + \tau_0|\nabla w|\right] d\omega - F_1 w|_{\partial\omega_1} - F_2 w|_{\partial\omega_2}$$

and also, certain singularities of the structure of function u_μ which minimizes $I_3(w)$. It was shown in [1] that when a cylinder of cross section ω moves in a viscoplastic medium under the action of force F, then the lengths of level lines $u_\mu = h$ of velocity distribution is defined by the equality

$$\tau_0 \int_\alpha^\beta \text{mes}\ \partial(u_\mu > h)\, dh \leqslant F(\beta - \alpha) \tag{23}$$

It follows from inequality (23) that function u_μ which minimises I_3, has a level line $u_\mu = u_0 + \rho(\mu)$ ($\rho \to 0$ when $\mu \to 0$) which separates the lines of level u_μ into two classes. Namely, for $u_\mu < u_0 + \rho(\mu)$ the level lines u_μ are simply-connected and contain region $\omega_1 \cup \omega_2$; for $u_\mu > u_0 + \rho(\mu)$ the level lines u_μ are doubly-connected and each of its components contains the related region ω_i $(i = 1,2)$. The boundary of the set of level $u_\mu = u_0 + \rho(\mu) = u_0^\mu$ is shown in Fig. 3 by a continuous line.

Let us consider regions T_1 through T_6 (Fig. 4) by using these lines. Straight lines AA_0 and BB_0 are perpendicular to $\partial\omega_1$; lines CC_0 and DD_0 are perpendicular to $\partial\omega_2$. Furthermore

$$A + \inf I_3(w) \geqslant \sum_1^6 \inf_{w_k} I_2(w_k, T_k) \tag{24}$$

$$I_2(\varphi, \Omega) = \int_\Omega \left[\frac{\mu}{2}|\nabla\varphi|^2 + \tau_0|\nabla\varphi|\right] d\omega$$

$$A = F_1 u_1|_{\partial\omega_1} + F_2 u_2|_{\partial\omega_2}$$

$$w_1|_{APB} = u_1, \quad w_2|_{AQB} = u_1, \quad w_2|_{A_0Q_0B_0} = u_0^\mu, \quad w_3|_{CLD} = u_2$$

$$w_3|_{C_0L_0D_0} = u_0^\mu, \quad w_4|_{CSD} = u_2, \quad w_5|_{A_0C_0} = u_0^\mu, \quad w_6|_{B_0D_0} = u_0^\mu$$

and w_1, w_4, w_5 and w_6 tend to vanish, when a point in the related region tends to infinity.

Using formula (15), we obtain

$$\inf_{w_1} I_2(w_1, T_1) \geqslant \tau_0 u_1 \operatorname{mes} APB + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0 u_1^3}\, M(\operatorname{mes} APB) + O(\mu) \tag{25}$$

$$\inf_{w_2} I_2(w_2, T_2) \geqslant \tau_0 (u_1 - u_0^\mu) \operatorname{mes} AQB + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_1 - u_0^\mu)^{3/2} M(\operatorname{mes} AQB) + O(\mu)$$

$$\inf_{w_3} I_2(w_3, T_3) \geqslant \tau_0 (u_2 - u_0^\mu) \operatorname{mes} CLD + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_2 - u_0^\mu)^{3/2} M(\operatorname{mes} CLD) + O(\mu)$$

$$\inf_{w_4} I_2(w_4, T_4) \geqslant \tau_0 u_2 \operatorname{mes} CSD + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0 u_2^3}\, M(\operatorname{mes} CSD) + O(\mu)$$

It is evident that the following relationships

$$I_2(w_5, T_5) \geqslant \tau_0 \int_{T_5} |\nabla w_5|\, d\omega = \tau_0 \int_0^{u_0} \operatorname{mes}[\partial(w_5 > h) \setminus \partial T_5]\, dh \geqslant \tau_0 u_0^\mu \operatorname{mes} AC \tag{26}$$

$$I_2(w_6, T_6) \geqslant \tau_0 u_0^\mu \operatorname{mes} BD \tag{27}$$

are valid. Finally, from formulas (24) – (27) we obtain the following lower bound estimate:

$$\begin{aligned} A + \inf I_3 \geqslant\ & \tau_0 u_1 \operatorname{mes} APB + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\, u_1^{3/2} M(\operatorname{mes} APB) + \\ & \tau_0 (u_1 - u_0^\mu) \operatorname{mes} AQB + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_1 - u_0^\mu)^{3/2} M(\operatorname{mes} AQB) + \\ & \tau_0 (u_2 - u_0^\mu) \operatorname{mes} CLD + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_2 - u_0^\mu)^{3/2} M(\operatorname{mes} CLD) + \\ & \tau_0 u_2 \operatorname{mes} CSD + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\, u_2^{3/2} M(\operatorname{mes} CSD) + \\ & \tau_0 u_0^\mu \operatorname{mes} AC + \tau_0 u_0^\mu \operatorname{mes} BP + O(\mu) \end{aligned} \tag{28}$$

To prove the impossibility of the inequality $u_0 < u_2$ it is sufficient to construct function v_μ for which $I_3(v_\mu) + A$ is strictly smaller than the right-hand part of inequality (28).

Let us consider the boundary layer of region ω_i.

The level line $u_\mu{}^* = u_2$ in the boundary layer of ω_1 is shown in Fig. 5 by the dash line. Its velocity field is determined by formula (10). Let us construct function v_μ. We draw tangents common to the dash line and $\partial\omega_2$ (Fig. 6). These tangents determine points A', C', D' and B' . The straight lines $A''A'$ and $C''C'$ are perpendicular to $A'C'$ and lines $B''B'$ and $D''D'$ are perpendicular to $B'D'$. Function v_μ is equal to the corresponding values in the boundary layers in regions ω_3, ω_4 and ω_5. In ω_6 and ω_7 the

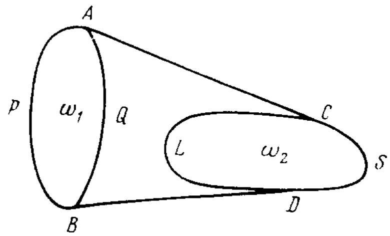

Fig. 1

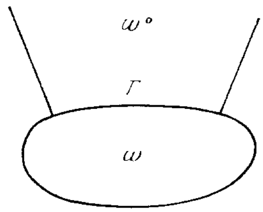

Fig. 2

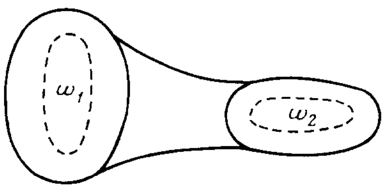

Fig. 3

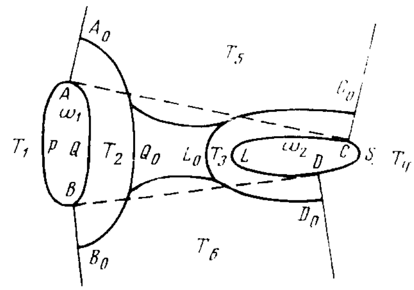

Fig. 4

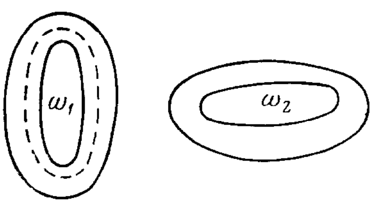

Fig. 5

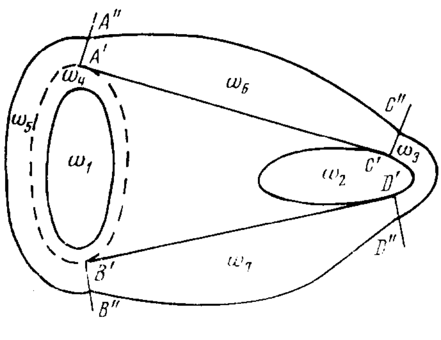

Fig. 6

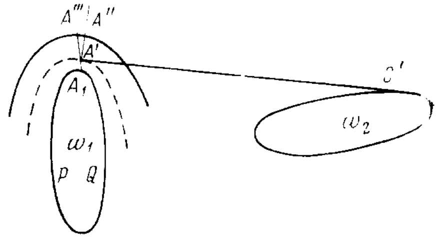

Fig. 7

lines of level v_μ are parabolas, with v_μ being a continuous function with piecewise-continuous derivatives. Direct computation will readily show that a set of parabolas such that

$$I_2(v_\mu, \omega_6 \cup \omega_7) = \tau_0 u_2 \operatorname{mes} A'C' + \tau_0 u_2 \operatorname{mes} B'D' + O(\sqrt{\mu}) \tag{29}$$

exists in regions ω_6 and ω_7

Note that, if relationship (29) is to be satisfied, the width of regions ω_6 and ω_7 must be of the order of $\mu^{1/2}$. Thus v_μ determines a motion which is possible from the point of view of kinematics in which part of the medium moves at the cylinder velocity ω_2.

Let us compute $A + I_3$ over function v_μ

$$A + I_3 = I_2(v_\mu, \omega_3) + I_2(v_\mu, \omega_4 \cup \omega_5) + O(\sqrt{\mu}) + \tau_0 u_2 \operatorname{mes} A'C' + \tau_0 u_2 \operatorname{mes} B'D' \tag{30}$$

The relationship

$$I_2(v_\mu, \omega_3) = \tau_0 u_2 \operatorname{mes} C'SD' + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\, u_2^{3/2} M(\operatorname{mes} C'SD') + O(\mu) \tag{31}$$

is valid by virtue of (15).

Let us consider the second integral in (20). We draw at points A' and B' (Fig. 6) normals to $\partial\omega_1$ (see Fig. 7). With the use of the form (10) of function v_μ we directly determine that

$$I_2(v_\mu, A'A''A''') = o(\sqrt{\mu}), \qquad I_2(v_\mu, B'B''B''') = o(\sqrt{\mu})$$

Hence the following formula

$$I_2(v_\mu, \omega_4 \cup \omega_5) = \tau_0 u_1 \operatorname{mes} A_1PB_1 + \tau_0(u_1 - u_2) \operatorname{mes} A_1QB_1 + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\, u_1^{3/2} M(\operatorname{mes} A_1PB_1) + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_1 - u_2)^{3/2} M(\operatorname{mes} A_1, QB_1) + o(\sqrt{\mu}) \tag{32}$$

is valid. From formulas (30) – (32) we obtain

$$\begin{aligned} A + I_3(v_\mu) = {} & \tau_0 u_1 \operatorname{mes} A_1PB_1 + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\, u_1^{3/2} M(\operatorname{mes} A_1PB_1) + \\ & \tau_0(u_1 - u_2) \operatorname{mes} A_1QB_1 + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_1 - u_2)^{3/2} M(\operatorname{mes} A_1QB_1) + \\ & \tau_0 u_2 \operatorname{mes} C'SD' + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\, u_2^{3/2} M(\operatorname{mes} C'SD') + \\ & \tau_0 u_2 \operatorname{mes} A'C' + \tau_0 u_2 \operatorname{mes} B'D' + O(\sqrt{\mu}) \end{aligned} \tag{33}$$

From formulas (28) and (33) we have

$$\begin{aligned} \inf I_3 - I_3(v_\mu) = {} & \tau_0 u_2 (\operatorname{mes} A_1QB_1 + \operatorname{mes} C'LD' - \\ & \operatorname{mes} A'C' - \operatorname{mes} B'D') + \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_2 - u_0)^{3/2} M(\operatorname{mes} CLD) + O(\sqrt{\mu}) \end{aligned} \tag{34}$$

In determining the right-hand part of (34) we used the relationship $A' \to A$, $B' \to B$, $C' \to C$ and $D' \to D$ when $\mu \to 0$. It can be readily shown that

$$\operatorname{mes} A_1QB_1 + \operatorname{mes} C'LD' - \operatorname{mes} A'C' - \operatorname{mes} B'D' \geqslant O(\sqrt{\mu}) \tag{35}$$

The inequality (35) follows from the relationship

$$\operatorname{mes} AQB + \operatorname{mes} CLD = \operatorname{mes} AC + \operatorname{mes} BD$$

and from the assumption that the curvature is positive at all points of $\partial\omega_1$ and $\partial\omega_2$.

Formulas (34) and (35) imply that

$$\inf I_3 - I_3(v_\mu) \geqslant \frac{2\sqrt{2}}{3}\sqrt{\mu\tau_0}\,(u_2-u_0)^{3/2} M\ (\mathrm{mes}\ CLD) + o(\sqrt{\mu}) \tag{36}$$

Inequality (36) shows that the assumption $u_0 < u_2$ is incorrect.

Thus the vanishing small viscosity selects from among the solutions of the perfectly plastic problem (2) that for which $u_0 = u_2$.

The derived principle of selecting the solution for a perfectly plastic medium is based in the considered problem on the assumption of positive curvature of the boundary $\partial\omega_i$. It is, however, apparent that this principle is also valid in the case of some relaxation of that assumption.

We note in conclusion that the problem of selecting a stationary solution for a perfectly plastic medium was considered in [2] from a different point of view. Although the allowance for inertial properties of the medium in the considered problem also results in the separation of a unique solution, the latter radically differs from the derived above. In this case only the cylinders move, while the medium remains stationary. A comparison of these two methods of selecting stationary solution shows that the separated solutions depend on the method of obtaining a strictly convex functional from the input one, taking into account that inertial and viscosity properties have different effects and that their combination may result in solutions of the intermediate kind, when passing to limit.

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